Mappings preserving unit distance on Heisenberg group

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Abstract

Let \( H^m \) be a Heisenberg group provided with a norm \( \rho \). A mapping \( f : H^m \to H^m \) is called preserving the distance \( n \) if for all \( x, y \) of \( H^m \) with \( \rho(x^{-1}y) = n \) then \( \rho(f(x)^{-1}f(y)) = n \). We obtain some results for the Aleksandrov problem in the Heisenberg group.

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1 Introduction

Posed around 1970, the Aleksandrov problem investigates an isometry by the preservation of some properties of distance [1]. Several studies have been established to this subject on different normed spaces. I quote in this connection, the studies made by H. Y. Chu, C. G. Park, W. G. Park in 2004 on linear 2-normed spaces [8], J. M. Rassias, S. Xiang, M. J. Rassias in 2007 on the Aleksandrov and triangle isometry Ulam stability problem [16], X.Y. Chen, M.M. Song in 2010 on linear n-normed spaces [7] and D. Wang, Y. Liu, M. Song in 2012 on non-Archimedean normed spaces [23]. For more details the reader may also study [[2]-[6], [9]-[13], [15, 17],[18]-[22]].

The purpose of our contribution, is an idea introduced by J. M. Rassias that consists to apply it here to study the Aleksandrov problem in a Heinsenberg group.

Following this Introduction, some preliminary notations are set in the second Section, as well as our main new results are investigated in the third Section, respectively.

2 Preliminary

In this section we fix notations and special vocabulary that will be used later in the document. Let \( m \) be a fixed nonzero integer number. The \( m^{th} \) Heisenberg group \( H^m \) is of course a near isomorphism \( \mathbb{C}^m \times \mathbb{R} \) endowed with the following group law

\[
(z, t)(z', t') = (z + z', t + t' + \text{Im} \langle z, z' \rangle), \quad (z, t), (z', t') \in \mathbb{C}^m \times \mathbb{R},
\]

where \( z = (z_i)_{1 \leq i \leq m} \), \( z' = (z'_i)_{1 \leq i \leq m} \) and \( \langle z, z' \rangle = \sum_{i=1}^{m} z_i \overline{z'_i} \), with identity element \((0, 0)\) and an inverse given by \((z, t)^{-1} = (-z, -t)\). The dilation \( \delta_s \), for \( s > 0 \), acts on the Heisenberg group as \( \delta_s(z, t) = (sz, s^2t) \) and is its automorphism. The homogeneous norm

\[
\rho(z, t) = (|z|^4 + t^2)^{\frac{1}{4}}
\]
defines the Heisenberg metric $d_\rho$ via the formula
\[ d_\rho(x, y) = \rho(x^{-1}y), \ x, y \in H^m. \]

Observe that the Heisenberg metric is really a metric and not just a quasi-metric since,
\[ \rho(xy) \leq \rho(x) + \rho(y) \]
for all $x, y \in H^m$ (see [7, 8] for instance). It is also known that the Heisenberg metric $d_\rho$ and the Carnot Caratheodory metric $d$ are equivalent; that is, there exists a constant $c > 1$ such that $c^{-1}d(x, y) \leq d_\rho(x, y) \leq cd(x, y)$ for all $x, y \in H^m$.

3 Main result
Let us establish in this section the main results of this paper. We note that, throughout this section $H^m$ designates a Heisenberg group with its norm $\rho$.

**Definition 3.1.** A mapping $f$ of $H^m$ on itself is called an isometry if
\[ \rho(f(x)^{-1}f(y)) = \rho(x^{-1}y) \]
for all $x, y \in H^m$.

If a mapping $f$ of $H^m$ on itself is an isometry then the inverse mapping is an isometry of $H^m$ onto $H^m$.

**Definition 3.2.** A mapping $f$ of $H^m$ on itself, satisfies the strong distance one preserving property (SDOPP) if and only if for all $x, y \in H^m$ with $\rho(x^{-1}y) = 1$ it follows that $\rho(f(x)^{-1}f(y)) = 1$.

**Definition 3.3.** A mapping $f : H^m \to H^m$ satisfies the strong distance $n$ preserving property (SDnPP) if only if for all $x, y \in H^m$ with $\rho(x^{-1}y) = n$ it follows that $\rho(f(x)^{-1}f(y)) = n$.

**Definition 3.4.** Let $H^m$ be a Heisenberg group. We call a mapping $f : H^m \to H^m$ Lipschitz mapping if there is a $K > 0$ such that
\[ \rho(f(x)^{-1}f(y)) \leq K\rho(x^{-1}y) \]
for any $x, y \in H^m$.

**Definition 3.5.** We call a mapping $f : H^m \to H^m$ locally Lipschitz mapping if there is a $K > 0$ such that
\[ \rho(f(x)^{-1}f(y)) \leq K\rho(x^{-1}y), \]
whenever $\rho(x^{-1}y) \leq 1$.

We consider in this paper only the Lipschitz constant $K \leq 1$.

In this paper we shall study, mappings satisfying the weaker assumption that they preserve distance $n$ in both directions, instead of isometries. We shall see that such mappings are not far from being isometries. Let us prove the following Lemma.

**Lemma 3.6.** Let $H^m$ the Heisenberg group. Suppose that $f : H^m \to H^m$ is a surjective mapping satisfying (SDOPP). Then $f$ is bijective.
Proof. We shall show that \( f \) is injective. Suppose that there exist \( x \) and \( y \) in \( H^m \) with \( x \neq y \) such that \( f(x) = f(y) \). Since \( \rho(x^{-1}y) \neq 0 \), we can set

\[
z = x \delta \frac{1}{\rho(y^{-1}x)} (y^{-1}x),
\]

thus,

\[
\rho(x^{-1}z) = \rho(\delta \frac{1}{\rho(y^{-1}x)} (y^{-1}x)) = 1.
\]

Since \( f \) verify \( f \) preserves the unit distance, that \( \rho(f(x)^{-1}f(z)) = 1 \). Now we will prove that \( \rho(y^{-1}z) \neq 1 \). Suppose on the contrary, that \( \rho(y^{-1}z) = 1 \). We have \( y^{-1}z = y^{-1}x \delta \frac{1}{\rho(y^{-1}x)} (y^{-1}x) \), by identification we can put \( y^{-1}x = (x_1, t_1) \) and denote \( \alpha = \frac{1}{\rho(y^{-1}x)} \). This implies that

\[
\rho(y^{-1}z) = \rho((1 + \alpha)x_1, (1 + \alpha^2)t_1) = ((1 + \alpha)^4|x_1|^4 + (1 + \alpha^2)^2 t_1^2)^\frac{1}{2}.
\]

Since \( \rho(y^{-1}z) = 1 \) and \( \alpha^4|x_1|^4 + \alpha^4 t_1^2 = 1 \), so

\[
(1 + 4\alpha^3 + 4\alpha^3 + 2\alpha^2)|x_1|^4 + (1 + 2\alpha^2)t_1^2 = 0
\]

and so \( x_1 = 0 \) and \( t_1 = 0 \), then \( x = y \), which is a contradiction. Therefore \( \rho(y^{-1}z) \neq 1 \). Now, we get \( 1 = \rho(f(x)^{-1}f(z)) = \rho(f(y)^{-1}f(z)) \). Since \( f \) preserves the unit distance, that \( \rho(y^{-1}z) = 1 \) which is a contradiction. Therefore \( f \) is a injective and surjective mapping then \( f \) is bijective.

Q.E.D.

The following theorem gives the \( n \)-distance preserving mapping in both directions

**Theorem 3.7.** Let \( H^m \) the Heisenberg group. Suppose that \( f : H^m \to H^m \) is a surjective mapping satisfying (SDOPP) such that

\[
\rho(x^{-1}y) < 1 \text{ if and only if } \rho(f(x)^{-1}f(y)) < 1.
\]

Then \( f \) preserves the area \( n \) for each \( n \in \mathbb{N} \).

Proof. By Lemma 3.6 \( f \) is a injective and since \( f \) is surjective mapping then \( f \) is bijective. Both \( f \) and \( f^{-1} \) preserves the unit distance and verify 3.1. Now we will prove that \( f \) preserves distance \( n \) in both directions for any positive integer \( n \). In the sequel we shall need the following notations:

\[
B(x; r) = \{ z : \rho(x^{-1}z) \leq r \};
\]

\[
B_0(x; r) = \{ z : \rho(x^{-1}z) < r \};
\]

Let \( x \) be an arbitrary vector in \( H^m \) and \( n \) any positive integer, \( n > 1 \). Assume that \( z \in B(x, n) \). We can find a sequence \( x = x_0, \ldots, z = x_n \), such that

\[
x_i = x_{i-1} \delta \frac{1}{\rho(x^{-1}x)} (z^{-1}x), \ i = 1, \ldots, n.
\]

Then,

\[
\rho(x_i^{-1}x_i) = \rho(\delta \frac{1}{\rho(x^{-1}x)} (z^{-1}x)) = 1, \ i = 1, \ldots, n.
\]
Since $f$ preserves the unit distance, that
\[ \rho(f(x)^{-1}f(z)) \leq n \]
Consequently, we have
\[ f(B(x, n)) \subset B(f(x), n), n > 1. \]
The same result can be obtained for $f^{-1}$. Hence,
\[ f(B(x, n)) = B(f(x), n), n > 1. \]

Since $f$ and $f^{-1}$ verify 3.1 then
\[ f(B_0(x, 1)) = B_0(f(x), 1). \]
hold for all $x \in H^m$. Now we will prove that
\[ f(B_0(x, n)) = B_0(f(x), n). \]
for all $x \in X$, and $n \in \mathbb{N}$. Let $z \in B_0(x, n)$ and consider a sequence $x = x_0, x_1, ..., x_n = z$ such that
\[ x_i = x_{i-1} \delta_1^{1/n}(z^{-1}x), \quad i = 1, ..., n. \]
Then
\[ \rho(x_{i-1}^{-1}x_i) = \rho(\delta_1^{1/n}(z^{-1}x)) = 1, \quad i = 1, ..., n. \]
Since $f(B_0(x, 1)) = B_0(f(x), 1)$, that
\[ \rho(f(x)^{-1}f(z)) < n \]
Consequently, we have
\[ f(B_0(x, n)) \subset B_0(f(x), n), n \in \mathbb{N}. \]
The same result can be obtained for $f^{-1}$. Hence,
\[ f(B_0(x, n)) = B_0(f(x), n), n \in \mathbb{N}. \]

Consequently
\[ f(B(x, n)) \setminus f(B_0(x, n)) = B(f(x), n) \setminus B_0(f(x), n), n \in \mathbb{N}. \]
So $f$ preserve the area $n$ for each $n \in \mathbb{N}$.

We will study the problem of mappings which preserve unit distance is an isometry.

**Lemma 3.8.** If a mapping $f$ is locally Lipschitz, then $f$ is a Lipschitz mapping.

**Proof.** We may assume that $\rho(x^{-1}y) \geq 1$, then there is $n_0 \in \mathbb{N}$ such that $\rho(x^{-1}y) \leq n_0$. Let $x = x_0 = (y_0, t_0), x_1, ..., x_{n_0} = y$, such that
\[ x_i = x_{i-1} \delta_1^{1/n_0}(y^{-1}x), \quad i = 1, ..., n_0. \]
Then
\[ \rho(x_i^{-1}x_i) = \rho(\delta_{\frac{1}{n_0}}(y^{-1}x)) \leq 1, \ i = 1, \ldots, n. \]

Since \( f \) is locally Lipschitz, so
\[ \rho(f(x_i^{-1}f(x_i)) \leq K\rho(x_i^{-1}x_i), \]

consequently
\[ \rho(f(x)^{-1}f(y)) \leq K\rho(x^{-1}y). \]

Q.E.D.

**Theorem 3.9.** Let \( f : H^m \to H^m \) be a locally Lipschitz mapping with the Lipschitz constant \( K \leq 1 \). Assume that \( f \) is a surjective mapping satisfying (SDOPP). Then \( f \) is a isometry.

**Proof.** By Lemma 3.6 and 3.8, \( f \) Lipschitz mapping with the Lipschitz constant \( K \leq 1 \) and \( f \) is bejective. For \( x, y \in X \), there are two cases depending upon whether \( \rho(f(x)^{-1}f(y)) = 0 \) or not.

In the first case \( \rho(f(x)^{-1}f(y)) = 0 \) equivalent to \( f(x) = f(y) \). Since \( f \) is injective so \( x = y \) and so \( \rho(x^{-1}y) = 0 \).

In the remaining case \( \rho(f(x)^{-1}f(y)) > 0 \), there is an \( n_0 \in \mathbb{N} \) such that \( \rho(f(x)^{-1}f(y)) < n_0 \). Assume that \( \rho(f(x)^{-1}f(y)) < \rho(x^{-1}y) \). Set \( x = x_0, x_1, \ldots, x_{n_0} = y, \) such that
\[ x_i = x_{i-1} \delta_{\frac{1}{n_0}}(f(y)^{-1}f(x)), \ i = 1, \ldots, n_0. \]

Then we obtain that
\[ \rho(x_i^{-1}x_i) = \frac{\rho(f(y)^{-1}f(x))}{n_0}, \ i = 1, \ldots, n_0. \]

Thus
\[ \rho(x^{-1}y) = \rho(\prod_{i=1}^{n_0} x_i^{-1}x_i) \]
\[ \leq \sum_{i=1}^{n_0} \rho(x_i^{-1}x_i) \]
\[ = \sum_{i=1}^{n_0} \rho(f(y)^{-1}f(x)) \]
\[ = \rho(f(x)^{-1}f(y)). \]

Which is a contradiction, hence \( f \) is a isometry. Q.E.D.

**References**


